SECTION 15.2: LIMITS AND CONTINUITY OF FUNCTIONS OF SEVERAL VARIABLES

WARM-UP QUESTION: What is $\lim\limits_{x \to a} f(x)$ asking for? How does this relate to $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$?

EXAMPLE 1: Let $f(x, y) = \dfrac{2y^2}{x^2 + y^2}$.

1. What is the domain of f?

Ans: $\{(x, y) \, : \, (x, y) \neq (0, 0)\}$

2. What do you think $\lim\limits_{(x,y) \to (0,0)} f(x, y)$ is asking for?

3. How many different ways can $(x, y) \to (0, 0)$?

4. Investigate $\lim\limits_{(x,y) \to (0,0)} f(x, y)$ along the following paths:

   Along the line $y = x$:

   Ans: 1

   Along the parabola $y = x^2$:

   Ans: 0

   Based on your calculations, would you say $\lim\limits_{(x,y) \to (0,0)} f(x, y)$ exists? Why or why not?

RECALL: $\lim_{x\to a} f(x) = L$ means given $\epsilon > 0$ there is a $\delta > 0$ so that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

QUESTION: Can you still draw the picture that goes with this definition and explain what it means?

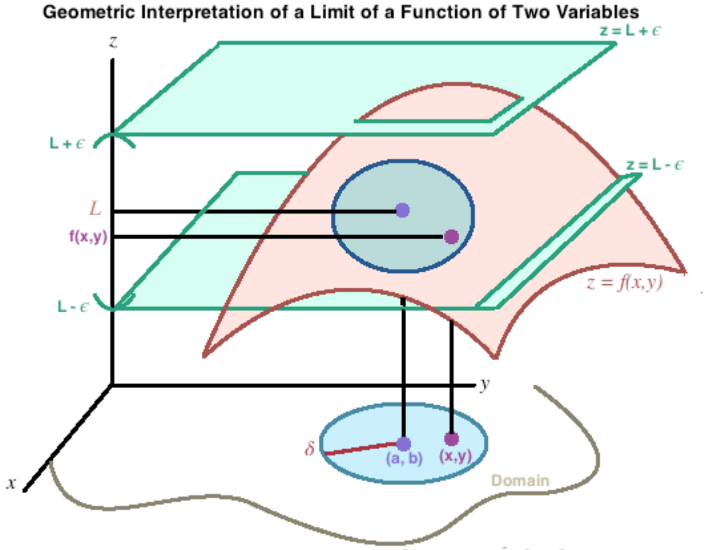
We can take this same idea and extend it to functions of two (or more) variables as follows. Note since the **inputs** to f are **points**, we use the **distance formula** in the plane to measure the distance between the inputs. Since the **outputs** are real **numbers**, we can stick with the absolute value to measure the distance between them.

DEFINITION: $\lim_{(x,y)\to(a,b)} f(x,y) = L$ means:

Given $\epsilon > 0$ there is a $\delta > 0$ so that if $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$, then $|f(x,y) - L| < \epsilon$.

In pictures:

Geometric Interpretation of a Limit of a Function of Two Variables



The limit as (x, y) approaches (a, b) is L if for all $\epsilon > 0$ there exists a $\delta > 0$ such that if (x,y) is in the domain of f and (x, y) is within $\delta > 0$ of (a,b), then the subset of points from the surface generated by the function f is contained between the two planes z = L + ϵ and z = L - ϵ.

NOTE: If $\lim_{(x,y)\to(a,b)} f(x,y) = L$, then $f(x,y)$ must approach L regardless of how points (x,y) approach (a,b).

TWO PATH TEST:

If two paths produce two **different** values for $\lim_{(x,y)\to(a,b)} f(x,y)$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does **not** exist.

NOTE:

If two paths produce the **same** value for $\lim_{(x,y)\to(a,b)} f(x,y)$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ **may** or **may not** exist.

EXAMPLE 2: Show that $\lim_{(x,y)\to(0,0)} \dfrac{2y^3}{x^2+y^2} = 0$ along all paths of the form $y = mx$.

Can we conclude $\lim_{(x,y)\to(0,0)} \dfrac{2y^3}{x^2+y^2} = 0$? Explain.

EXAMPLE 3: Prove $\lim_{(x,y)\to(0,0)} \dfrac{2y^3}{x^2+y^2} = 0$ using the definition of limit.

LIMIT LAWS

Since the definition of limit here is the same idea as before, all of the usual limit laws hold.

PROPERTIES OF LIMITS: Suppose $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ and $\lim\limits_{(x,y)\to(a,b)} g(x,y) = M$:

- **CONSTANT LAW:** For all constants c, $\lim\limits_{(x,y)\to(a,b)} c = c$
- **IDENTITY LAWS:** $\lim\limits_{(x,y)\to(a,b)} x = a$ and $\lim\limits_{(x,y)\to(a,b)} y = b$
- **CONSTANT MULTIPLE:** For all constants c, $\lim\limits_{(x,y)\to(a,b)} c\,f(x,y) = c \lim\limits_{(x,y)\to(a,b)} f(x,y) = c\,L$
- **SUM / DIFFERENCE:** $\lim\limits_{(x,y)\to(a,b)} [f(x,y) \pm g(x,y)] = \lim\limits_{(x,y)\to(a,b)} f(x,y) \pm \lim\limits_{(x,y)\to(a,b)} g(x,y) = L \pm M$
- **PRODUCT RULE:** $\lim\limits_{(x,y)\to(a,b)} [f(x,y)g(x,y)] = \left[\lim\limits_{(x,y)\to(a,b)} f(x,y)\right]\left[\lim\limits_{(x,y)\to(a,b)} g(x,y)\right] = L\,M$
- **QUOTIENT RULE:** If $M \neq 0$: $\lim\limits_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{\lim_{(x,y)\to(a,b)} f(x,y)}{\lim_{(x,y)\to(a,b)} g(x,y)} = \dfrac{L}{M}$
- **POWER RULE:** For natural numbers n, $\lim\limits_{(x,y)\to(a,b)} [f(x,y)]^n = \left[\lim\limits_{(x,y)\to(a,b)} f(x,y)\right]^n = L^n$
- **ROOT RULE:** If $\sqrt[n]{L}$ is a real number, $\lim\limits_{(x,y)\to(a,b)} \sqrt[n]{f(x,y)} = \sqrt[n]{\lim\limits_{(x,y)\to(a,b)} f(x,y)} = \sqrt[n]{L}$

EXAMPLE 4: Find $\lim\limits_{(x,y)\to(0,2)} \dfrac{3y\cos(2x)}{\sqrt{x^2+y^2}}$

Ans: $\lim\limits_{(x,y)\to(0,2)} \dfrac{3y\cos(2x)}{\sqrt{x^2+y^2}} = 3$

NOTE: As seen above, the **Substitution Principle** from single-variable Calculus carries over to multivariable!

If direct substitution results in an indeterminate form, we can often algebraically manipulate the function into a form where we can apply the limit laws. That is:

THEOREM: If $f(x, y) = g(x, y)$ for all (x, y) except (a, b), and if $\lim\limits_{(x,y)\to(a,b)} g(x, y) = L$, then $\lim\limits_{(x,y)\to(a,b)} f(x, y) = L$.

EXAMPLE 5: Find the following limits if they exist.

1. $\lim\limits_{(x,y)\to(2,-1)} \dfrac{x^2 - 4y^2}{x + 2y}$

Ans: $\lim\limits_{(x,y)\to(2,-1)} \dfrac{x^2 - 4y^2}{x + 2y} = 4$

2. $\lim\limits_{(x,y)\to(-2,-2)} \dfrac{\sqrt{xy} - 2}{xy - 4}$

Ans: $\lim\limits_{(x,y)\to(-2,-2)} \dfrac{\sqrt{xy} - 2}{xy - 4} = \dfrac{1}{4}$

3. $\lim\limits_{(x,y)\to(-1,3)} \dfrac{\dfrac{1}{x+y} - \dfrac{1}{2}}{x + y - 2}$

Ans: $\lim\limits_{(x,y)\to(-1,3)} \dfrac{\dfrac{1}{x+y} - \dfrac{1}{2}}{x + y - 2} = -\dfrac{1}{4}$

POLAR COORDINATES: A helpful substitution to try for indeterminate forms as $(x, y) \to (0, 0)$ is to use polar coordinates: $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Then $(x, y) \to (0, 0)$ is equivalent to $r \to 0$.

EXAMPLE 6: Prove $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2y^3}{x^2+y^2} = 0$ by converting to Polar Coordinates.

Ans: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2y^3}{x^2+y^2} = \lim_{r\to 0} 2r\sin^3(\theta) = 0$

As we'll see, there is no multivariable version of L'Hopital's Rule . . . we do still get the Squeeze Theorem, however:

SQUEEZE THEOREM:

If $h(x, y) \le f(x, y) \le g(x, y)$ for all (x, y) in a disk centered at (a, b) except possibly at (a, b) and if $\displaystyle\lim_{(x,y)\to(a,b)} h(x, y) = \lim_{(x,y)\to(a,b)} g(x, y) = L$, then $\displaystyle\lim_{(x,y)\to(a,b)} f(x, y) = L$

EXAMPLE 7: Use the Squeeze Theorem to determine $\displaystyle\lim_{(x,y)\to(1,0)} y\cos\left(\frac{x}{y}\right)$.

Ans: $\displaystyle\lim_{(x,y)\to(1,0)} y\cos\left(\frac{x}{y}\right) = 0$

CONTINUITY OF FUNCTIONS OF SEVERAL VARIABLES

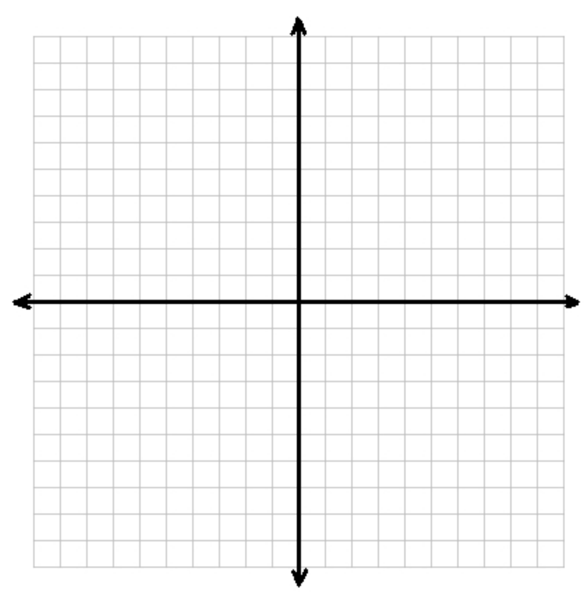
RECALL: f is **continuous** at $x = a$ if $\lim_{x\to a} f(x) = f(a)$. That is, f is continuous at $x = a$ if:

- $f(a)$ exists.
- $\lim_{x\to a} f(x)$ exists.
- $\lim_{x\to a} f(x) = f(a)$.

EXAMPLE 8: Generalize the above definition: a function $f(x, y)$ is continuous at $(x, y) = (a, b)$ means:

Since the limit laws are the same in multivariable Calculus as they are in single-variable Calculus, continuous functions behave as expected. Namely **combinations of continuous functions are continuous** and for the most part, **functions are continuous on their domains.**

EXAMPLE 9: Sketch the **regions** of continuity of $f(x, y) = \sqrt{xy}$ below.



EXAMPLE 10: Extend the definition of $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ to three variables.

That is, define: $\lim\limits_{(x,y,z)\to(a,b,c)} F(x,y,z) = L$.

EXAMPLE 11: Find $\lim\limits_{(x,y,z)\to(1,2,3)} \dfrac{\cos(\pi xz)}{y\ln(x)+z^2}$

Ans: $\lim\limits_{(x,y,z)\to(1,2,3)} \dfrac{\cos(\pi xz)}{y\ln(x)+z^2} = -\dfrac{1}{9}$

EXAMPLE 12: Define what it means for a function $F(x,y,z)$ to be continuous at a point (a,b,c).

HOMEWORK: Section 15.2: 13 - 77 every other odd, 79, 81